Distinguished representations and exceptional poles of the Asai-L-function

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Abstract

Let K/F be a quadratic extension of p-adic fields. We show that a generic irreducible representation of $GL(n,K)$ is distinguished if and ony if its Rankin-Selberg Asai L-function has an exceptional pole at zero. We use this result to compute Asai L-functions of ordinary irreducible representations of $GL(2,K)$. In the appendix, we describe supercuspidal dihedral representations of $GL(2,K)$ in terms of Langlands parameter.

Introduction

For K/F a quadratic extension of local fields, let σ be the conjugation relative to this extension, and $\eta_{K/F}$ be the character of F^* whose kernel is the set of norms from K^*. The conjugation σ extends naturally to an automorphism of $GL(n,K)$, which we also denote by σ. If π is a representation of $GL(n,K)$, we denote by π^σ the representation $g \mapsto \pi(\sigma(g))$.
If π is a smooth irreducible representation of $GL(n,K)$, and χ a character of F^*, the dimension of the space of linear forms on its space, which transform by χ under $GL(n,F)$ (with respect to the action $[(L,g) \mapsto L \circ \pi(g)]$), is known to be at most one (Proposition 11, [F1]). One says that π is χ-distinguished if this dimension is one, and says that π is distinguished if it is 1-distinguished.
Jacquet conjectured two results about distinguished representations of $GL(n,K)$. Let π be a smooth irreducible representation of $GL(n,K)$ and $\pi^\vee$ its contragredient. The first conjecture states that it is equivalent for π with central character trivial on F^* to be isomorphic to $\pi^{\vee\sigma}$ and for π to be distinguished or $\eta_{K/F}$-distinguished. In [K], Kable proved it for discrete series representations, using Asai L-functions.
The second conjecture, which is proved in [K], states that if π is a discrete series representation, then it cannot be distinguished and $\eta_{K/F}$-distinguished at the same time.
One of the key points in Kable's proof is that if a discrete series representation of $GL(n,K)$ is such that its Asai L-function has a pole at zero, then it is distinguished, Theorem 1.4 of [A-K-T] shows that it is actually an equivalence. This theorem actually shows that Asai L-functions of tempered distinguished representations admit a pole at zero.
In this article, using a result of Youngbin Ok which states that for a distinguished representation, linear forms invariant under the affine subgroup of $GL(n,F)$ are actually $GL(n,F)$-invariant (which

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generalises Corollary 1.2 of [A-K-T]), we prove in Theorem 2.1 that a generic representation is distinguished if and only if its Asai L-function admits an exceptional pole at zero. A pole at zero is always exceptional for Asai L-functions of discrete series representations (see explanation before Proposition 2.4). As a first application, we give in Proposition 2.6 a formula for Asai L-functions of supercuspidal representations of $GL(n, K)$.
There are actually three different ways to define Asai L-functions: one via the local Langlands correspondence and in terms of Langlands parameters denoted by $L_W(\pi, s)$, the one we use via the theory of Rankin-Selberg integrals denoted by $L_{As}(\pi, s)$, and the Langlands-Shahidi method applied to a suitable unitary group, denoted by $L_{As,2}(\pi, s)$ (see [A-R]). It is expected that the above three L-functions are equal.
For a discrete series representation π, it is shown in [He] that $L_W(\pi, s) = L_{As,2}(\pi, s)$, and in [A-R] that $L_{As}(\pi, s) = L_{As,2}(\pi, s)$, both proofs using global methods.
As a second application of our principal result, we show (by local methods) in Theorem 3.2 of Section 3 that for an ordinary representation (i.e. corresponding through Langlands correspondence to an imprimitive 2 dimensional representation of the Weil-Deligne group) π of $GL(2, K)$, we have $L_W(\pi, s) = L_{As}(\pi, s)$. We recall that for odd residual characteristic, every smooth irreducible infinite-dimensional representation of $GL(2, K)$ is ordinary.
In the appendix (Section 4), we describe in Theorem 4.4 distinguished dihedral supercuspidal representations, this description is used in Section 3 for the computation of L_{As} for such representations.

1 Preliminaries

Let E_1 be a field, and E_2 a finite galois extension of E_1, we denote by $Gal(E_2/E_1)$ the Galois group of E_2 over E_1, and we denote by Tr_{E_2/E_1} (respectively N_{E_2/E_1}) the trace (respectively the norm) function from E_2 to E_1. If E_2 is quadratic over E_1, we denote by σ_{E_2/E_1} the non trivial element of $Gal(E_2/E_1)$.
In the rest of this paper, the letter F will always designate a non archimedean local field of characteristic zero in a fixed algebraic closure $\bar{F}$, and the letter K a quadratic extension of F in $\bar{F}$. We denote by q_F and q_K the cardinality of their residual fields, R_K and R_F their integer rings, P_K and P_F the maximal ideals of R_K and R_F, and U_K and U_F their unit groups. We also denote by v_K and v_F the respective normalized valuations, and $| \ |_K$ and $| \ |_F$ the respective absolute values. We fix an element δ of $K - F$ such that $\delta^2 \in F$, hence $K = F(\delta)$.
Let ψ be a non trivial character of K trivial on F, it is of the form $x \mapsto \psi' \circ Tr_{K/F}(\delta x)$ for some non trivial character ψ' of F.
Whenever G is an algebraic group defined over F, we denote by $G(K)$ its K-points and $G(F)$ its F-points. The group $GL(n)$ is denoted by G_n, its standard maximal unipotent subgroup is denoted by N_n.
If π is a representation of a group, we also denote by π its isomorphism class. Let μ be a character of F^*, we say that a representation π of $G_n(K)$ is μ-distinguished if it admits on its space V_π a linear form L, which verifies the following: for v in V and h in $G_n(K)$, then $L(\pi(h)v) = \mu(det(h))L(v)$. If $\mu = 1$, we say that π is distinguished.
We denote by $K_n(F)$ the maximal compact subgroup $G_n(R_F)$ of $G_n(F)$, and for $r \geq 1$, we denote by $K_{n,r}(F)$, the congruence subgroup $I_n + M_n(P_F^r)$.
The character ψ defines a character of $N_n(K)$ that we still denote by ψ, given by $\psi(n) = \psi(\sum_{i=1}^{n-1} n_{i,i+1})$.

We now recall standard results from [F2].
Let π be a generic smooth irreducible representation of $G_n(K)$, we denote by $\pi^\vee$ its smooth contragredient, and c_π its central character.
We denote by $D(F^n)$ the space of smooth functions with compact support on F^n, and $D_0(F^n)$ the subspace of $D(F^n)$ of functions vanishing at zero. We denote by ρ the natural action of $G_n(F)$ on $D(F^n)$, given by $\rho(g)\phi(x_1,\dots,x_n) = \phi((x_1,\dots,x_n)g)$, and we denote by η the row vector $(0,\dots,0,1)$ of length n.
If W belongs to the Whittaker model $W(\pi,\psi)$ of π, and ϕ belongs to $D(F^n)$, the following integral converges for s of real part large enough:

$$\int_{N_n(F)\backslash G_n(F)} W(g)\phi(\eta g)|det(g)|_F{}^s dg.$$

This integral as a function of s has a meromorphic extension to $\mathbb{C}$ which we denote by $\Psi(W,\phi,s)$. For s of real part large enough, the function $\Psi(W,\phi,s)$ is a rational function in q_F^{-s}, which actually has a Laurent series development.
The $\mathbb{C}$-vector space generated by these functions is in fact a fractional ideal $I(\pi)$ of $\mathbb{C}[q_F^{-s},q_F^s]$. This ideal $I(\pi)$ is principal, and has a unique generator of the form $1/P(q_F^{-s})$, where P is a polynomial with $P(0)=1$.

Definition 1.1. *We denote by $L_{As}(\pi,s)$ the generator of $I(\pi)$ defined just above, and call it the Asai L-function of π.*

Remark 1.1. If P belongs to $\mathbb{C}[X]$ and has constant term equal to one, then the function of the complex variable $L_P : s \mapsto 1/P(q_F^{-s})$ is called an Euler factor. It is a meromorphic function on $\mathbb{C}$ and admits $(2i\pi/ln(q_F))\mathbb{Z}$ as a period subgroup. Hence if s_0 is a pole of L_P, the elements $s_0 + (2i\pi/ln(q_F))\mathbb{Z}$ are also poles of L_P, with same multiplicities, we identify s_0 and $s_0+(2i\pi/ln(q_F))\mathbb{Z}$ when we talk about poles. A pole s_0 then corresponds to a root α_0 of P by the formula $q^{-s_0}=\alpha_0$, its multiplicity in L_P equal to the multiplicity of α_0 in P.

Let w_n be the matrix of $G_n(\mathbb{Z})$ with ones on the antidiagonal, and zeroes elsewhere. For W in $W(\pi,\psi)$, we denote by $\tilde{W}$ the function $g \mapsto W(w_n{}^t g^{-1})$ which belongs to $W(\pi^\vee,\psi^{-1})$, and we denote by $\widehat{\phi}$ the Fourier transform (with respect to ψ' and its associate autodual Haar measure) of ϕ in $D(F^n)$.

Theorem 1.1. ***(Functional equation)****(Th. of [F2])*
There exists an epsilon factor $\epsilon_{As}(\pi,s,\psi)$ which is, up to scalar, a (maybe negative) power of q^s, such that the following functional equation is satisfied for any W in $W(\pi,\psi)$ and any ϕ in $D(F^n)$:

$$\Psi(\tilde{W},\widehat{\phi},1-s)/L_{As}(\pi^\vee,1-s) = c_\pi(-1)^{n-1}\epsilon_{As}(\pi,s,\psi)\Psi(W,\phi,s)/L_{As}(\pi,s).$$

We finally recall the following, which will be crucial in the demonstration of Theorem 2.1.

Proposition 1.1. *([Ok], Theorem 3.1.2) Let π be an irreducible distinguished representation of $G_n(K)$, if L is a $P_n(F)$-invariant linear form on the space of π, then it is actually $G_n(F)$-invariant.*

Sketch of the proof. We note V the space of π, and $\tilde{V}$ that of $\pi^\vee$. As the representation $\pi^\vee$ is isomorphic to $g \mapsto \pi((g^t)^{-1})$, it is also distinguished. Let L be a $P_n(F)$-invariant linar form on

the space V and $\tilde{L}$ a $G_n(F)$-invariant linar form on the space $\tilde{V}$, the linear form $L \otimes \tilde{L}$ on $V \otimes \tilde{V}$ is $P_n(F) \times G_n(F)$-invariant. It is thus enough to prove that a linear form B on $V \otimes \tilde{V}$ which is $P_n(F) \times G_n(F)$-invariant is $G_n(F) \times G_n(F)$-invariant.
Call λ the (right) action by left translation and ρ that by right translation of $G_n(K)$ on the space $C_c^\infty(G_n(K))$, it follows from Lemma p.73 of [B] that there exists an injective morphism I of $G_n(K) \times G_n(K)$-modules from $[(\pi \otimes \pi^\vee)^*, (V \otimes \tilde{V})^*]$ to $[(\lambda \times \rho)^*, (C_c^\infty(G_n(K)))^*]$. The linear form $I(B)$ is an element of $(C_c^\infty(G_n(K)))^*$ which is $P_n(F) \times G_n(F)$-invariant. As I is injective, the result will follow from the fact that an invariant distribution on $G_n(K)/G_n(F)$ which is invariant by left translation under $P_n(F)$ is actually $G_n(F)$-invariant. Identifying $G_n(K)/G_n(F)$ with the space S of matrices g of $G_n(K)$ verifying of $gg^\sigma = 1$ (see [S], ch.10, prop.3), this statement is exactly the one of Lemma 5 of [G-J-R]. $\square$

2 Poles of the Asai L-function and distinguishedness

Now suppose $L_{As}(\pi, s)$ has a pole at s_0, its order d is the highest order pole of the family of functions of $I(\pi)$.
Then we have the following Laurent expansion at s_0:

$$\Psi(W, \phi, s) = B_{s_0}(W, \phi)/(q_F^s - q_F^{s_0})^d + \textit{ smaller order terms.} \tag{1}$$

The residue $B_{s_0}(W, \phi)$ defines a non zero bilinear form on $W(\pi, \psi) \times D(F^n)$, satisfying the quasi-invariance:

$$B_{s_0}(\pi(g)W, \rho(g)\phi) = |det(g)|_F^{-s_0} B_{s_0}(W, \phi).$$

Following [C-P] for the split case $K = F \times F$, we state the following definition:

Definition 2.1. *A pole of the Asai L-function $L_{As}(\pi, s)$ at s_0 is called exceptional if the associated bilinear form B_{s_0} vanishes on $W(\pi, \psi) \times D_0(F^n)$.*

As an immediate consequence, if s_0 is an exceptional pole of $L_{As}(\pi, s)$, then B_{s_0} is of the form $B_{s_0}(W, \phi) = \lambda_{s_0}(W)\phi(0)$, where λ_{s_0} is a non zero $|det(\ )|_F^{-s_0}$ invariant linear form on $W(\pi, \psi)$. Hence we have:

Proposition 2.1. *Let π be a generic irreducible representation of $G_n(K)$, and suppose its Asai L-function has an exceptional pole at zero, then π is distinguished.*

We denote by $P_n(F)$ the affine subgroup of $G_n(F)$, given by matrices with last row equal to η. For more convenience, we introduce a second L-function: for W in $W(\pi, \psi)$, by standard arguments, the following integral is convergent for $Re(s)$ large, and defines a rational function in q^{-s}, which has a Laurent series development:

$$\int_{N_n(F)\backslash P_n(F)} W(p)|det(p)|_F{}^s dp.$$

We denote by $\Psi_1(W, s)$ the corresponding Laurent series. By standard arguments again, the vector space generated by the functions $\Psi_1(W, s-1)$, for W in $W(\pi, \psi)$, is a fractional ideal $I_1(\pi)$ of $\mathbb{C}[q_F^{-s}, q_F^s]$, which has a unique generator of the form $1/Q(q_F^{-s})$, where Q is a polynomial with $Q(0) = 1$. We denote by $L_1(\pi, s)$ this generator.

Lemma 2.1. *([J-P-S] p. 393)*
Let W be in $W(\pi,\psi)$, one can choose ϕ with support small enough around $(0,\dots,0,1)$ such that $\Psi(W,\phi,s)=\Psi_1(W,s-1)$.

Proof. As we gave a reference, we only sketch the proof. We first recall the following integration formula (cf. proof of the proposition in paragraph 4 of [F]), for $Re(s)>>0$:

$$\Psi(W,\phi,s)=\int_{K_n(F)}\int_{N_n(F)\backslash P_n(F)} W(pk)|det(p)|_F^{s-1}dp\int_{F^*}\phi(\eta ak)c_\pi(a)|a|_F^{ns}d^*adk. \qquad (2)$$

Choosing r large enough for W to be right invariant under $K_{n,r}(F)$, we take ϕ a positive multiple of the characteristic function of $\eta K_{n,r}(F)$, and conclude from equation (2). □

Hence we have the inclusion $I_1(\pi)\subset I(\pi)$, which implies that $L_1(\pi,s)=L_{As}(\pi,s)R(q_F^s,q_F^{-s})$ for some R in $\mathbb{C}[q_F^{-s},q_F^s]$. But because L_1 and L_{As} are both Euler factors, R is actually just a polynomial in q_F^{-s}, with constant term equal to one. Noting $L_{rad(ex)}(\pi,s)$ its inverse (which is an Euler factor), we have $L_{As}(\pi,s)=L_1(\pi,s)L_{rad(ex)}(\pi,s)$, we will say that L_1 divides L_{As}. The explanation for the notation $L_{rad(ex)}$ is given in Remark 2.1.

We now give a characterisation of exceptional poles:

Proposition 2.2. *A pole of $L_{As}(\pi,s)$ is exceptional if and only if it is a pole of the function $L_{rad(ex)}(\pi,s)$ defined just above.*

Proof. From equation (2), it becomes clear that the vector space generated by the integrals $\Psi(W,\phi,s)$ with W in $W(\pi,\psi)$ and ϕ in $D_0(F^n)$, is contained in $I_1(\pi)$, but because of Lemma 2.1, those two vector spaces are equal. Hence $L_1(\pi,s)$ is a generator of the ideal generated as a vector space by the functions $\Psi(W,\phi,s)$ with W in $W(\pi,\psi)$ and ϕ in $D_0(F^n)$.
From equation (1), if s_0 is an exceptional pole, a function $\Psi(W,\phi,s)$, with ϕ in $D_0(F^n)$, cannot have a pole of highest order at s_0, hence we have one implication.
Now if the order of the pole s_0 for $L_{As}(\pi,s)$ is stricly greater than the one of $L_1(\pi,s)$, then the first residual term corresponding to a pole of highest order of the Laurent development of any function $\Psi(W,\phi,s)$ with $\phi(0)=0$ must be zero, and zero is exceptional. □

Lemma 2.1 also implies:

Proposition 2.3. *The functional $\Lambda_{\pi,s}:W\mapsto\Psi_1(W,s-1)/L_{As}(\pi,s)$ defines a (maybe null) linear form on $W(\pi,\psi)$ which transforms by $|det(\ )|_F^{1-s}$ under the affine subgroup $P_n(F)$.*
For fixed W in $W(\pi,\psi)$, then $s\mapsto\Lambda_{\pi,s}(W)$ is a polynomial of q_F^{-s}.

Now we are able to prove the converse of Proposition 2.1:

Theorem 2.1. *A generic irreducible representation π of $G_n(K)$ is distinguished if and only if $L_{As}(s,\pi)$ admits an exceptional pole at zero.*

Proof. We only need to prove that if π is distinguished, then $L_{As}(s,\pi)$ admits an exceptional pole at zero, so we suppose π distinguished.
From equation (2), for $Re(s)<<0$, and π distinguished (so that c_π has trivial restriction to F^*), one has:

$$\Psi(\tilde{W},\widehat{\phi},1-s)=\int_{K_n(F)}\int_{N_n(F)\backslash P_n(F)}\tilde{W}(pk)|det(p)|_F^{-s}dp\int_{F^*}\widehat{\phi}(\eta ak)|a|_F^{n(1-s)}d^*adk. \quad (3)$$

This implies that:

$$\Psi(\tilde{W},\widehat{\phi},1-s)/L_{As}(\pi^\vee,1-s)=\int_{K_n(F)}\Lambda_{\pi^\vee,1-s}(\pi^\vee(k)\tilde{W})\int_{F^*}\widehat{\phi}(\eta ak)|a|_F^{n(1-s)}d^*adk. \quad (4)$$

The second member of the equality is actually a finite sum: $\sum_i \lambda_i\Lambda_{\pi^\vee,1-s}(\pi^\vee(k_i)\tilde{W})\int_{F^*}\widehat{\phi}(\eta ak_i)|a|_F^{n(1-s)}d^*a$, where the λ_i's are positive constants and the k_i's are elements of $K_n(F)$ independant of s.
Note that there exists a positive constant ϵ, such that for $Re(s)<\epsilon$, the integral $\int_{F^*}\widehat{\phi}(\eta ak_i)|a|_F^{n(1-s)}d^*a$ is absolutely convergent, and defines a holomorphic function. So we have an equality (equality 4) of analytic functions (actually of polynomials in q_F^{-s}), hence it is true for all s such that $Re(s)<\epsilon$.
For $s=0$, we get:

$$\Psi(\tilde{W},\widehat{\phi},1)/L_{As}(\pi^\vee,1)=\int_{K_n(F)}\Lambda_{\pi^\vee,1}(\pi^\vee(k)\tilde{W})\int_{F^*}\widehat{\phi}(\eta ak)|a|_F^n d^*adk.$$

But as π is distinguished, so is $\pi^\vee$, and as $\Lambda_{\pi^\vee,1}$ is a $P_n(F)$-invariant linear form on $W(\pi^\vee,\psi^{-1})$, it follows from Propodsition 1.1 that it is actually $G_n(F)$-invariant.
Finally

$$\Psi(\tilde{W},\widehat{\phi},1)/L_{As}(\pi^\vee,1)=\Lambda_{\pi^\vee,1}(\tilde{W})\int_{K_n(F)}\int_{F^*}\widehat{\phi}(\eta ak)|a|_F^n d^*adk$$

which is equal to:

$$\Lambda_{\pi^\vee,1}(\tilde{W})\int_{P_n(F)\backslash G_n(F)}\widehat{\phi}(\eta g)d_\mu g$$

where d_μ is up to scalar the unique $|det(\ )|^{-1}$ invariant measure on $P_n(F)\backslash G_n(F)$. But as

$$\int_{P_n(G)\backslash G_n(F)}\widehat{\phi}(\eta g)d_\mu g=\int_{F^n}\widehat{\phi}(x)dx=\phi(0),$$

we deduce from the functional equation that $\Psi(W,\phi,0)/L_{As}(\pi,0)=0$ whenever $\phi(0)=0$.
As one can choose W, and ϕ vanishing at zero, such that $\Psi(W,\phi,s)$ is the constant function equal to 1 (see the proof of Theorem 1.4 in [A-K-T]), hence $L_{As}(\pi,s)$ has a pole at zero, which must be exceptional. $\square$

For a discrete series representation π, it follows from Lemma 2 of [K], that the integrals of the form

$$\int_{N_n(F)\backslash P_n(F)}W(p)|det(p)|_F{}^{s-1}dp.$$

converge absolutely for $Re(s)>-\epsilon$ for some positive ϵ, hence as functions of s, they cannot have a pole at zero.
This implies that $L_1(\pi,s)$ has no pole at zero, hence Theorem 2.1 in this case gives:

Proposition 2.4. *([K], Theorem 4)*

A discrete series representation π of $G_n(K)$ is distinguished if and only if $L_{As}(s,\pi)$ admits a pole at zero.

Let s_0 be in $\mathbb{C}$. We notice that if π is a generic irreducible representation of $G_n(K)$, it is $|\ |_F^{-s_0}$-distinguished if and only if $\pi \otimes |\ |_K^{s_0/2}$ is distinguished, but as $L_{As}(s, \pi \otimes |\ |_K^{s_0/2})$ is equal to $L_{As}(s+s_0,\pi)$, Theorem 2.1 becomes:

Theorem 2.2. *A generic irreducible representation π of $G_n(K)$ is $|\ |_F^{-s_0}$-distinguished if and only if $L_{As}(s,\pi)$ admits an exceptional pole at s_0.*

Remark 2.1. Let P and Q be two polynomials in $\mathbb{C}[X]$ with constant term 1, we say that the Euler factor $L_P(s) = 1/P(q_F^{-s})$ divides $L_Q(s) = 1/Q(q_F^{-s})$ if and only P divides Q. We denote by $L_P \vee L_Q$ the Euler factor $1/(P\vee Q)(q_F^{-s})$, where the l.c.m $P \vee Q$ is chosen such that $(P\vee Q)(0) = 1$. We define the g.c.d $L_P \wedge L_Q$ the same way.
It follows from equation (2) that if $c_{\pi|F^*}$ is ramified, then $L_{As}(\pi,s) = L_1(\pi,s)$. It also follows from the same equation that if $c_{\pi|F^*} = |\ |_F^{-s_1}$ for some s_1 in $\mathbb{C}$, then $L_{rad(ex)}(\pi,s)$ divides $1/(1-q_F^{s_1-ns})$. Anyway, $L_{rad(ex)}(\pi,s)$ has simple poles.
Now we can explain the notation $L_{rad(ex)}$. We refer to [C-P] where the case $K = F\times F$ is treated. In fact, in the latter, $L_{ex}(\pi,s)$ is the function $1/P_{ex}(\pi,q_F^{-s})$, with $P_{ex}(\pi,q_F^{-s}) = \prod_{s_i}(1-q_F^{s_i-s})^{d_i}$, where the s_i's are the exceptional poles of $L_{As}(\pi,s)$ and the d_i's their order in $L_{As}(\pi,s)$. Hence $L_{rad(ex)}(\pi,s) = 1/P_{rad(ex)}(\pi,q_F^{-s})$, where $P_{rad(ex)}(\pi,X)$ is the unique generator with constant term equal to one, of the radical of the ideal generated by $P_{ex}(\pi,X)$ in $\mathbb{C}[X]$.

We proved:

Proposition 2.5. *Let π be an irreducible generic representation of $G_n(K)$, the Euler factor $L_{rad(ex)}(\pi,s)$ has simple poles, it is therefore equal to $\prod 1/(1-q_F^{s_0-s})$ where the product is taken over the $q_F^{s_0}$'s such that π is $|\ |_F^{-s_0}$-distinguished.*

Suppose now that π is supercuspidal, then the restriction to $P_n(K)$ of any W in $W(\pi,\psi)$ has compact support modulo $N_n(K)$, hence $\Psi_1(W,s-1)$ is a polynomial in q^{-s}, and $L_1(\pi,s)$ is equal to 1. Hence Proposition 2.5 becomes:

Proposition 2.6. *Let π be an irreducible supercuspidal representation of $G_n(K)$, then $L_{As}(\pi,s) = \prod 1/(1-q^{s_0-s})$ where the product is taken over the q^{s_0}'s such that π is $|\ |_F^{-s_0}$-distinguished.*

3 Asai L-functions of $GL(2)$

3.1 Asai L-functions for imprimitive Weil-Deligne representations of dimension 2

The aim of this paragraph is to compute $L_W(\rho,s)$ (see the introduction) when ρ is an imprimitive two dimensional representation of the Weil-Deligne group of K.

We denote by W_K (resp. W_F) the Weil group of K (resp. F), I_K (resp. I_F) the inertia subgroup of W_K (resp. W_F), W_K' (resp. W_F') the group $W_K\times SL(2,\mathbb{C})$ (resp. $W_F\times SL(2,\mathbb{C})$) and I_K' (resp.

I'_F) the group $I_K \times SL(2,\mathbb{C})$ (resp. $I_F \times SL(2,\mathbb{C})$). We denote by ϕ_F a Frobenius element of W_F, and we also denote by ϕ'_F the element (ϕ_F, I_2) of W'_F.
We denote by $sp(n)$ the unique (up to isomorphism) complex irreducible representation of $SL(2,\mathbb{C})$ of dimension n.
If ρ is a finite dimensional representation of W'_K, we denote by $M_{W'_K}^{W'_F}(\rho)$ the representation of W'_F induced multiplicatively from ρ. We recall its definition:
If V is the space of ρ, then the space of $M_{W'_K}^{W'_F}(\rho)$ is $V \otimes V$. Noting τ an element of $W_F - W_K$, and σ the element (τ, I) of W'_F, we have:

$$M_{W'_K}^{W'_F}(\rho)(h)(v_1 \otimes v_2) = \rho(h)v_1 \otimes \rho^\sigma(h)v_2$$

for h in W'_K, v_1 and v_2 in V.

$$M_{W'_K}^{W'_F}(\rho)(\sigma)(v_1 \otimes v_2) = \rho(\sigma^2)v_2 \otimes v_1$$

for v_1 and v_2 in V.

We refer to paragraph 7 of [P] for definition and basic properties of multiplicative induction in the general case.

Definition 3.1. *The function $L_W(\rho, s)$ is by definition the usual L-function of the representation $M_{W'_K}^{W'_F}(\rho)$, i.e. $L_W(\rho, s) = L(M_{W'_K}^{W'_F}(\rho), s)$.*

i) If ρ is of the form $Ind_{W'_B}^{W'_K}(\omega)$ for some multiplicative character ω of a biquadratic extension B of F, we denote by K' and K'' the two other extensions between F and B. If we call σ_1 an element of W'_K which is not in $W'_{K'} \cup W'_{K''}$ and σ_3 an element of $W'_{K''}$ which is not in $W'_K \cup W'_{K'}$, then $\sigma_2 = \sigma_3\sigma_1$ is an element of $W'_{K'}$ which is not in $W'_K \cup W'_{K''}$.
The elements $(1, \sigma_1, \sigma_2, \sigma_3)$ are representatives of W'_F/W'_B, and 1 and σ_3 are representatives of W'_F/W'_K.
If one identifies ω with a character (still called ω) of B^*, then ω^{σ_1} identifies with $\omega \circ \sigma_{B/K}$, ω^{σ_2} with $\omega \circ \sigma_{B/K'}$ and ω^{σ_3} with $\omega \circ \sigma_{B/K''}$. One then verifies that if a belongs to W_B, one has:
• $Tr[M_{W'_K}^{W'_F}(\rho)(a)] = Tr[Ind_{W'_{K'}}^{W'_F}(M_{W'_B}^{W'_{K'}}(\omega))(a)] + Tr[Ind_{W'_{K''}}^{W'_F}(M_{W'_B}^{W'_{K''}}(\omega))(a)] = \omega\omega^{\sigma_2} + \omega\omega^{\sigma_3} + \omega^{\sigma_1}\omega^{\sigma_2} + \omega^{\sigma_1}\omega^{\sigma_3}$.
• $Tr[M_{W'_K}^{W'_F}(\rho)(\sigma_1 a)] = Tr[Ind_{W'_{K'}}^{W'_F}(M_{W'_B}^{W'_{K'}}(\omega))(\sigma_1 a)] + Tr[Ind_{W'_{K''}}^{W'_F}(M_{W'_B}^{W'_{K''}}(\omega))(\sigma_1 a)] = 0$.
• $Tr[M_{W'_K}^{W'_F}(\rho)(\sigma_2 a)] = Tr[Ind_{W'_{K'}}^{W'_F}(M_{W'_B}^{W'_{K'}}(\omega))(\sigma_2 a)] + Tr[Ind_{W'_{K''}}^{W'_F}(M_{W'_B}^{W'_{K''}}(\omega))(\sigma_2 a)] = \omega(\sigma_2 a\sigma_2 a) + \omega^{\sigma_1}(\sigma_2 a\sigma_2 a)$.
• $Tr[M_{W'_K}^{W'_F}(\rho)(\sigma_3 a)] = Tr[Ind_{W'_{K'}}^{W'_F}(M_{W'_B}^{W'_{K'}}(\omega))(\sigma_3 a)] + Tr[Ind_{W'_{K''}}^{W'_F}(M_{W'_B}^{W'_{K''}}(\omega))(\sigma_3 a)] = \omega(\sigma_3 a\sigma_3 a) + \omega^{\sigma_1}(\sigma_3 a\sigma_3 a)$.
Hence we have the isomorphism

$$M_{W'_K}^{W'_F}(\rho) \simeq Ind_{W'_{K'}}^{W'_F}(M_{W'_B}^{W'_{K'}}(\omega)) \oplus Ind_{W'_{K''}}^{W'_F}(M_{W'_B}^{W'_{K''}}(\omega)).$$

From this we deduce that

$$\boxed{L(M_{W'_K}^{W'_F}(\rho), s) = L(\omega_{|K'^*}, s) L(\omega_{|K''^*}, s).}$$

ii) Let L be a quadratic extension of F, such that $\rho = Ind_{W'_L}^{W'_K}(\chi)$, with χ regular, is not isomorphic to a representation of the form $Ind_{W'_B}^{W'_K}(\omega)$ as in i), then

$$\boxed{L(M_{W'_K}^{W'_F}(\rho), s) = 1.}$$

Indeed, we show that $M_{W'_K}^{W'_F}(\rho)^{I'_F} = \{0\}$. If it wasn't the case, the representation $(M_{W'_K}^{W'_F}(\rho), V)$ would admit a I'_F-fixed vector, and so would its contragredient V^*. Now in the subspace of I'_F-fixed vectors of V^*, choosing an eigenvector of $M_{W'_K}^{W'_F}(\rho)(\phi_F)$, we would deduce the existence of a linear form L on $(M_{W'_K}^{W'_F}(\rho), V)$ which transforms under W'_F by an unramified character μ of W'_F. If we identify μ with a character μ' of F^*, the restriction of μ to W'_K corresponds to $\mu' \circ N_{K/F}$ of K^*, so we can write it as $\theta\theta^\sigma$, where θ is a character of W'_K corresponding to an extension of μ' to K^*. As the restriction of $M_{W'_K}^{W'_F}$ to W'_K is isomorphic to $\rho \otimes \rho^\sigma$, we deduce that $\theta^{-1}\rho \otimes (\theta^{-1}\rho)^\sigma$ is W'_K distinguished, that is $\theta\rho^\vee \simeq (\theta^{-1}\rho)^\sigma$. But from the proof of Theorem 4.2, this would imply that $\theta^{-1}\rho$ hence ρ, could be induced from a character of a biquadratic extension of F, which we supposed is not the case.

iii) Suppose $\rho = sp(2)$ acts on the space $\mathbb{C}^2$ with canonical basis (e_1, e_2) by the natural action $\rho\,[h, M]\,(v) = M(v)$ for h in W_K, M in $SL(2,\mathbb{C})$ and v in $\mathbb{C}^2$. Then the space of $M_{W'_K}^{W'_F}(\rho)$ is $V \otimes V$ and $SL(2,\mathbb{C})$ acts on it as $sp(2) \otimes sp(2)$. Decomposing $V \otimes V$ as the direct sum $Alt(V) \oplus Sym(V)$, we see that $SL(2,\mathbb{C})$ acts as 1 on $Alt(V)$, and $M_{W'_K}^{W'_F}(\rho)\left[1, \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\right](e_1 \otimes e_1) = x^2 e_1 \otimes e_1$. Hence the representation of $SL(2,\mathbb{C})$ on $Sym(V)$ must be $sp(3)$. The Weil group W_F acts as $\eta_{K/F}$ on $Alt(V)$ and trivially on $Sym(V)$, finally $M_{W'_K}^{W'_F}(\rho)$ is isomorphic to $sp(3) \oplus \eta_{K/F}$. Tensoring with a character χ, we have $M_{W'_K}^{W'_F}(\chi sp(2)) = \chi_{|F^*} M_{W'_K}^{W'_F}(sp(2)) = \chi_{|F^*}\eta_{K/F} \oplus \chi_{|F^*} sp(3)$. Hence one has the following equality:

$$\boxed{L(M_{W'_K}^{W'_F}(\chi sp(2)), s) = L(\chi_{|F^*}\eta_{K/F}, s) L(\chi_{|F^*}, s+1).}$$

iv) If $\rho = \lambda \oplus \mu$, with λ and μ two characters of W'_K, then from [P], Lemma 7.1, we have $M_{W'_K}^{W'_F}(\rho) = \lambda_{|F^*} \oplus \mu_{|F^*} \oplus Ind_{W'_K}^{W'_F}(\lambda\mu^\sigma)$. Hence we have

$$\boxed{L(M_{W'_K}^{W'_F}(\rho)) = L(\lambda_{|F^*}, s) L(\mu_{|F^*}, s) L(\lambda\mu^\sigma, s).}$$

3.2 Asai L-functions for ordinary representations of $GL(2)$

In this subsection, we compute Asai L-functions for ordinary (i.e. non exceptional) representations of $G_2(K)$, and prove (Theorem 3.2) that they are equal to the corresponding functions L_W of imprimitive representations of W'_K.

In order to compute L_{As}, we first compute L_1, but this latter computation is easy because Kirillov models of infinite-dimensional irreducible representations of $G_2(K)$ are well-known (see [Bu], Th. 4.7.2 and 4.7.3).
Let π be an irreducible infinite-dimensional (hence generic) representation of $G_2(K)$, we have the following situations for the computation of $L_1(\pi, s)$.

i) and ii) If π is supercuspidal, its Kirillov model consists of functions with compact support on K^*, hence

$$L_1(\pi, s) = 1.$$

iii) If $\pi = \sigma(\chi)$ ($\sigma(\chi| \;|_K{}^{1/2}, \chi| \;|_K{}^{-1/2})$ in [Bu]) is a special series representation of $G_2(K)$, twist of the Steinberg representation by the character χ of K^*, the Kirillov model of π consists of functions of $D(K)$ multiplied by $\chi| \;|_K$. Hence their restrictions to F are functions of $D(F)$ multiplied by $\chi| \;|_F{}^2$, and the ideal $I_1(\pi)$ is generated by functions of s of the form

$$\int_{F^*} \phi(t)\chi(t)|t|_F{}^{s-1}|t|_F{}^2 d^*t = \int_{F^*} \phi(t)\chi(t)|t|_F{}^{s+1} d^*t,$$

for ϕ in $D(F)$, hence we have

$$L_1(\pi, s) = L(\chi_{|F^*}, s+1).$$

iv) If $\pi = \pi(\lambda, \mu)$ is the principal series representation (λ and μ being two characters of K^*, with $\lambda\mu^{-1}$ different from $| \;|$ and $| \;|^{-1}$) corresponding to the representation $\lambda \oplus \mu$ of W'_K.

If $\lambda \neq \mu$, the Kirillov model of π is given by functions of the form $| \;|_K{}^{1/2}\chi\phi_1 + | \;|_K{}^{1/2}\mu\phi_2$, for ϕ_1 and ϕ_2 in $D(K)$, and

$$L_1(\pi, s) = L(\lambda_{|F^*}, s) \vee L(\mu_{|F^*}, s).$$

If $\lambda = \mu$, the Kirillov model of π is given by functions of the form $| \;|_K{}^{1/2}\lambda\phi_1 + | \;|_K{}^{1/2}\lambda v_K(t)\phi_2$, for ϕ_1 and ϕ_2 in $D(K)$, and

$$L_1(\pi, s) = L(\lambda_{|F^*}, s)^2.$$

In order to compute $L_{rad(ex)}$ for ordinary representations, we need to know when they are distinguished by a character $| \;|_F^{-s_0}$ for some s_0 in $\mathbb{C}$, we will then use Theorem 2.2. The answer is given by the following, which is a mix of Theorem 4.4 and Proposition B.17 of [F-H]:

Theorem 3.1. a) *A dihedral supercuspidal representation π of $G_2(K)$ is $| \;|_F^{-s_0}$-distinguished if and only if there exists a quadratic extension B of K, biquadratic over F (hence there are two other extensions between F and B that we call K' and K''), and a character of B^* regular with respect to $N_{B/K}$ which restricts either to K' as $| \;|_{K'}^{-s_0}$ or to K'' as $| \;|_{K''}^{-s_0}$, such that π is equal to $\pi(\omega)$.*

b) *Let μ be a character of K^*, then the special series representation $\sigma(\mu)$ is $|\ |_F^{-s_0}$-distinguished if and only if μ restricts to F^* as $\eta_{K/F}|\ |_F^{-s_0}$.*

c) *Let λ and μ be two characters of K^*, with $\lambda\mu^{-1}$ and $\lambda^{-1}\mu$ different from $|\ |_K$, then the principal series representation $\pi(\lambda,\mu)$ is $|\ |_F^{-s_0}$-distinguished if and only if either λ and μ restrict as $|\ |_F^{-s_0}$ to F^* or $\lambda\mu^\sigma$ is equal to $|\ |_K^{-s_0}$.*

Proof. Let π be a representation, it is $|\ |_F^{-s_0}$-distinguished if and only if $\pi \otimes |\ |_K^{s_0/2}$ is distinguished because $|\ |_K^{-s_0/2}$ extends $|\ |_F^{-s_0}$, it then suffices to apply Theorem 4.4 and Proposition B.17 of [F-H]. We give the full proof for case a). Suppose π is dihedral supercuspidal and $\pi\otimes|\ |_K^{s_0/2}$ is distinguished. From Theorem 4.4, the representation $\pi \otimes |\ |_K^{s_0/2}$ must be of the form $\pi(\omega)$, for ω a character of quadratic extension B of K, biquadratic over F, such that if we call K' and K'' two other extensions between F and B, ω doesn't factorize through $N_{B/K}$ and restricts either trivially on K'^*, or trivially on K''^*. But π is equal to $\pi(\omega)\otimes|\ |_K^{-s_0/2} = \pi(\omega|\ |_B^{-s_0/2})$ because $|\ |_B = |\ |_K \circ N_{B/K}$. As $|\ |_B^{-s_0/2}$ restricts to K' (resp. K'') as $|\ |_{K'}^{-s_0}$ (resp. $|\ |_{K''}^{-s_0}$), case a) follows. □

We are now able to compute $L_{rad(ex)}$, hence L_{As} for ordinary representations.

i) Suppose that $\pi = \pi(Ind_{W_B'}^{W_K'}(\omega)) = \pi(\omega)$ is supercuspidal, with Langlands parameter $Ind_{W_B'}^{W_K'}(\omega)$, where ω is a multiplicative character of a biquadratic extension B over F that doesn't factorize through $N_{B/K}$.
We denote by K' and K'' the two other extensions between B and F. Here $L_1(\pi,s)$ is equal to one.
We have the following series of equivalences:

$$\begin{aligned} s_0 \textit{ is a pole of } L_{As}(\pi(\omega),s) &\Longleftrightarrow \pi(\omega) \textit{ is } |\ |_F^{-s_0} - \textit{distinguished} \\ &\Longleftrightarrow \omega_{|K'^*} = |\ |_{K'}^{-s_0} \textit{ or } \omega_{|K''^*} = |\ |_{K''}^{-s_0} \\ &\Longleftrightarrow s_0 \textit{ is a pole of } L(\omega_{|K'^*},s) \textit{ or of } L(\omega_{|K''^*},s) \\ &\Longleftrightarrow s_0 \textit{ is a pole of } L(\omega_{|K'^*},s) \vee L(\omega_{|K''^*},s) \end{aligned}$$

As both functions $L_{As}(\pi(\omega),s)$ and $L(\omega_{|K'^*},s) \vee L(\omega_{|K''^*},s)$ have simple poles and are Euler factors, they are equal. Now suppose that $L(\omega_{|K'^*},s)$ and $L(\omega_{|K''^*},s)$ have a common pole s_0, this would imply that $\omega_{|K'^*} = |\ |_{K'}^{-s_0}$ and $\omega_{|K''^*} = |\ |_{K''}^{-s_0}$, which would mean that $\omega|\ |_B^{s_0/2}$ is trivial on $K'^*K''^*$. According to Lemma 4.2, this would contradict the fact that ω does not factorize through $N_{B/K}$, hence $L(\omega_{|K'^*},s) \vee L(\omega_{|K''^*},s) = L(\omega_{|K'^*},s)L(\omega_{|K''^*},s)$. Finally we proved:

$$\boxed{L_{As}(\pi(\omega),s) = L(\omega_{|K'^*},s)L(\omega_{|K''^*},s).}$$

ii) Suppose that π is a supercuspidal representation, corresponding to an imprimitive representation of W_K' that cannot be induced from a character of the Weil-Deligne group of a biquadratic extension of F. Then necessarily π cannot be $|\ |_F^{-s_0}$-distinguished, for any complex number s_0 of $\mathbb{C}$.
If it was the case, from Theorem 3.1, it would correspond to a Weil representation $\pi(\omega)$

for some multiplicative character of a biquadratic extension of F, which cannot be. Hence $L_{rad(ex)}(\pi, s)$ has no pole and is equal to one because it is an Euler factor, so we proved that:

$$\boxed{L_{As}(\pi, s) = 1.}$$

iii) If π is equal to $\sigma(\chi)$, then $L_1(\pi, s) = L(\chi_{|F^*}, s+1)$. We want to compute $L_{rad(ex)}(\pi, s)$, we have the following series of equivalences:

$$\begin{aligned} s_0 \textit{ is an exceptional pole of } L_{As}(\sigma(\chi), s) &\iff \sigma(\chi) \textit{ is } | \, |_F^{-s_0} - \textit{distinguished} \\ &\iff \chi_{|F^*} = \eta_{K/F} | \, |_F^{-s_0} \\ &\iff s_0 \textit{ is a pole of } L(\chi_{|F^*}\eta_{K/F}, s) \end{aligned}$$

As both functions $L_{rad(ex)}(\pi, s)$ and $L(\chi_{|F^*}\eta_{K/F}, s)$ have simple poles and are Euler factors, they are equal, we thus have:

$$\boxed{L_{As}(\sigma(\chi) = L(\chi_{|F^*}, s+1)L(\chi_{|F^*}\eta_{K/F}, s).}$$

iv) If $\pi = \pi(\lambda, \mu)$, we first compute $L_{rad(ex)}(\pi, s)$. We have the following series of equivalences:

$$\begin{aligned} s_0 \textit{ is an exceptional pole of } L_{As}(\pi(\lambda, \mu), s) &\iff \pi(\lambda, \mu) \textit{ is } | \, |_F^{-s_0} - \textit{distinguished} \\ &\iff \lambda\mu^\sigma = | \, |_K^{-s_0} \textit{ or, } \lambda_{|F^*} = | \, |_F^{-s_0} \textit{ and } \mu_{|F^*} = | \, |_F^{-s_0} \\ &\iff s_0 \textit{ is a pole of } L(\lambda\mu^\sigma, s) \textit{ or of } L(\lambda_{|F^*}, s) \wedge L(\mu_{|F^*}, s) \\ &\iff s_0 \textit{ is a pole of } L(\lambda\mu^\sigma, s) \vee [L(\lambda_{|F^*}, s) \wedge L(\mu_{|F^*}, s)] \end{aligned}$$

As both functions $L_{rad(ex)}(\pi(\lambda, \mu), s)$ and $L(\lambda\mu^\sigma, s) \vee [L(\lambda_{|F^*}, s) \wedge L(\mu_{|F^*}, s)]$ have simple poles and are Euler factors, they are equal.

If $\lambda \neq \mu$, then $L_1(\pi, s) = L(\lambda_{|F^*}, s) \vee L(\mu_{|F^*}, s)$. But $L(\lambda\mu^\sigma, s)$ and $L(\lambda_{|F^*}, s) \wedge L(\mu_{|F^*}, s)$ have no common pole. If there was a common pole s_0, one would have $\lambda\mu^\sigma = | \, |_K^{-s_0}$, $\lambda_{|F^*} = | \, |_F^{-s_0}$ and $\mu_{|F^*} = | \, |_F^{-s_0}$. From $\mu_{|F^*} = | \, |_F^{-s_0}$, we would deduce that $\mu \circ N_{K/F} = | \, |_K^{-s_0}$, i.e. $\mu^\sigma = | \, |_K^{-s_0}\mu^{-1}$, and $\lambda\mu^\sigma = | \, |_K^{-s_0}$ would imply $\lambda = \mu$, which is absurd. Hence $L_{rad(ex)}(\pi, s) = L(\lambda\mu^\sigma, s)[L(\lambda_{|F^*}, s) \wedge L(\mu_{|F^*}, s)]$, and finally we have $L_{As}(\pi, s) = L_1(\pi, s)L_{rad(ex)}(\pi, s) = L(\lambda_{|F^*}, s)L(\mu_{|F^*}, s)L(\lambda\mu^\sigma, s)$.
If λ is equal to μ, then $L_1(\pi, s) = L(\lambda_{|F^*}, s)^2$, and $L_{rad(ex)}(\pi(\lambda, \mu), s) = L(\lambda \circ N_{K/F}, s) \vee L(\lambda_{|F^*}, s)$. As $L(\lambda \circ N_{K/F}, s) = L(\lambda_{|F^*}, s)L(\eta_{K/F}\lambda_{|F^*}, s)$, we have $L_{rad(ex)}(\pi(\lambda, \mu), s) = L(\lambda \circ N_{K/F}, s)$. Again we have $L_{As}(\pi, s) = L(\lambda_{|F^*}, s)L(\mu_{|F^*}, s)L(\lambda\mu^\sigma, s)$.
In both cases, we have

$$\boxed{L_{As}(\pi(\lambda, \mu), s) = L(\lambda_{|F^*}, s)L(\mu_{|F^*}, s)L(\lambda\mu^\sigma, s).}$$

Eventually, comparing with equalities of subsection 3.1, we proved the following:

Theorem 3.2. *Let $\rho \mapsto \pi(\rho)$ be the Langlands correspondence from two dimensional representations of W'_K to smooth irreducible infinite-dimensional representations of $G_2(K)$, then if ρ is not primitive, $\pi(\rho)$ is ordinary and we have the following equality of L-functions:*

$$\boxed{L_{As}(\pi(\rho), s) = L(M_{W'_K}^{W'_F}(\rho), s)}$$

As said in the introduction, combining Theorem 1.6 of [A-R] and Theorem of pargraph 1.5 in [He], one gets that $L(M_{W'_K}^{W'_F}(\rho), s) = L_{As}(\pi(\rho), s)$ for $\pi(\rho)$ a discrete series representation, so that we have actually the following:

Theorem 3.3. *Let $\rho \mapsto \pi(\rho)$ be the Langlands correspondence from two dimensional representations of W'_K to smooth irreducible infinite-dimensional representations of $G_2(K)$, we have the following equality of L-functions:*

$$\boxed{L_{As}(\pi(\rho), s) = L(M_{W'_K}^{W'_F}(\rho), s)}$$

Conclusion . The results of Section 3 give a local proof of the equality of L_W and L_{As}, and effective computations of these functions. As it was said in the introduction, the latter equality is known for discrete series representations of $G_n(K)$ but the proof is global. Hence the essentially new information is the equality for principal series representations of $G_2(K)$.
Now the following conjecture is expected to be true:

Conjecture 3.1. *Let $(n_1, \ldots, n_t)$ be a partition of n, and for each i between 1 and t, let Δ_i be a quasi-square-integrable representation of $G_{n_i}(K)$. The generic representation π of $G_n(K)$ obtained by normalised parabolic induction of the Δ_i's is distinguished if and only if there is a reordering of these representations and an integer r between 1 and $t/2$, such that $\Delta_{i+1}^{\sigma} = \Delta_i^{\vee}$ for $i = 1, 3, .., 2r-1$, and Δ_i is distiunguished for $i > 2r$.*

In a work to follow, we intend to prove that assuming this conjecture, the functions L_W and L_{As} agree on generic representations of $G_n(K)$. As Conjecture 3.1 is proved in [M] for principal series representations, this would give the equality of the L functions for principal series representations of $G_n(K)$.

4 Appendix. Dihedral supercuspidal distinguished representations

The aim of this section is to give a description of dihedral supercuspidal distinguished representations of $G_2(K)$ in terms of Langlads parameter, it is done in Theorem 4.4.

4.1 Preliminary results

Let E be a local field, E' be a quadratic extension of E, χ a character of E^*, π be a smooth irreducible infinite-dimensional representation of $G_2(E)$, and ψ a non trivial character of E.
We denote by $L(\chi, s)$ and $\epsilon(\chi, s, \psi)$ the functions of the complex variable s defined before Proposition 3.5 in [J-L].We denote by $\gamma(\chi, s, \psi)$ the ratio $\epsilon(\chi, s, \psi)L(\chi, s)/L(\chi^{-1}, 1-s)$.
We denote by $L(\pi, s)$ and $\epsilon(\pi, s, \psi)$ the functions of the complex variable s defined in Theorem 2.18 of [J-L]. We denote by $\gamma(\pi, s, \psi)$ the ratio $\epsilon(\pi, s, \psi)L(\pi, s)/L(\pi^{\vee}, 1-s)$.
We denote by $\lambda(E'/E, \psi)$ the Langlands-Deligne factor defined before Proposition 1.3 in [J-L], it is equal to $\epsilon(\eta_{E'/E}, 1/2, \psi)$. As $\eta_{E'/E}$ is equal to $\eta_{E'/E}^{-1}$, the factor $\lambda(E'/E, \psi)$ is also equal to

$\gamma(\eta_{E'/E}, 1/2, \psi)$.
From Theorem 4.7 of [J-L], if ω is a character of E'^*, then $L(\pi(\omega), s)$ is equal to $L(\omega, s)$, and $\epsilon(\pi, s, \psi)$ is equal to $\lambda(E'/E, \psi)\epsilon(\pi, s, \psi)$, hence $\gamma(\pi, s, \psi)$ is equal to $\lambda(E'/E, \psi)\gamma(\pi, s, \psi)$.

We will need four results. The first is due to Fröhlich and Queyrut, see [D] Theorem 3.2 for a quick proof using a Poisson formula:

Proposition 4.1. *Let E be a local field, E' be a quadratic extension of E, χ' a character of E'^* trivial on E^*, and ψ' a non trivial character of E' trivial on E, then $\gamma(\chi', 1/2, \psi') = 1$.*

The second is a criterion of Hakim:

Theorem 4.1. *([Ha], Theorem 4.1) Let π be an irreducible supercuspidal representation of $G_2(K)$ with central character trivial on F^*, and ψ a nontrivial character of K trivial on F. Then π is distinguished if and only if $\gamma(\pi \otimes \chi, 1/2, \psi) = 1$ for every character χ of K^* trivial on F^*.*

The third is due to Flicker:

Theorem 4.2. *([F1], Proposition 12) Let π be a smooth irreducible distinguished representation of $G_n(K)$, then π^σ is isomorphic to $\pi^\vee$.*

The fourth is due to Kable in the case of $G_n(K)$, see [A-T] for a local proof in the case of $G_2(K)$:

Theorem 4.3. *([A-T], Proposition 3.1 There exists no supercuspidal representation of $G_2(K)$ which is distinguished and $\eta_{K/F}$-distinguished at the same time.*

4.2 Distinction criterion for dihedral supercuspidal representations

As a dihedral representation's parameter is a multiplicative character of a quadratic extension L of K, we first look at the properties of the tower $F \subset K \subset L$. Three cases arise:

1. L/F is biquadratic (hence Galois), it contains K and two other quadratic extensions F, K' and K''.

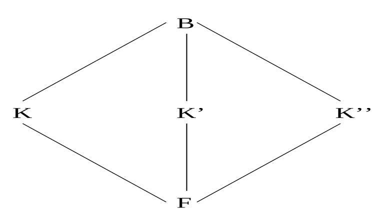

Figure 1:

Its Galois group is isomorphic with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, its non trivial elements are $\sigma_{L/K}$, $\sigma_{L/K'}$ and $\sigma_{L/K''}$. The conjugation $\sigma_{L/K}$ extend $\sigma_{K'/F}$ and $\sigma_{K''/F}$.

2. L/F is cyclic with Galois group isomorphic with $\mathbb{Z}/4\mathbb{Z}$, in this case we fix an element $\tilde{\sigma}$ in $G(L/F)$ extending σ, it is of order 4.

3. L/F non Galois. Then its Galois closure M is quadratic over L and the Galois group of M over F is dihedral with order 8. To see this, we consider a morphism $\tilde{\theta}$ from L to $\bar{F}$ which extends θ. Then if $L' = \tilde{\theta}(L)$, L and L' are distinct, quadratic over K and generate M biquadratic over K. M is the Galois closure of L because any morphism from L into $\bar{F}$, either extends θ, or the identity map of K, so that its image is either L or L', so it is always included in M. Finally the Galois group M over F cannot be abelian (for L is not Galois over F), it is of order 8, and it's not the quaternion group which only has one element of order 2, whereas here $\sigma_{M/L}$ and $\sigma_{M/L'}$ are of order 2. Hence it is the dihedral group of order 8 and we have the folowing lattice, where M/K' is cyclic of degree 4, M/K and B/F are biquadratic.

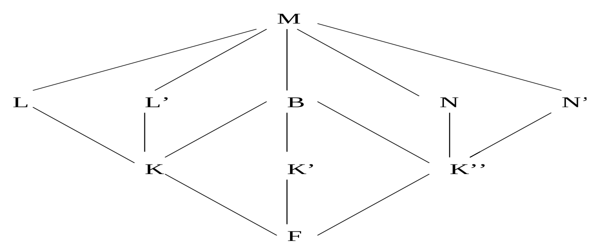

Figure 2:

We now prove the following proposition:

Proposition 4.2. *If a supercuspidal dihedral representation π of $G_2(K)$ verifies $\pi^\vee = \pi^\sigma$, there exists a biquadratic extension B of F, containing K, such that if we call K' and K'' the two other extensions between F and B, there is a character ω of B trivial either on $N_{B/K'}(B^*)$ or on $N_{B/K''}(B^*)$, such that $\pi = \pi(\omega)$.*

Proof. Let L be a quadratic extension of K and ω a regular multiplicative of L such that $\pi = \pi(\omega)$, we denote by σ the conjugation of L over K, three cases show up:

1. L/F is biquadratic. The conjugations $\sigma_{L/K'}$ and $\sigma_{L/K''}$ both extend σ, hence from Theorem 1 of [G-L], we have $\pi(\omega)^\sigma = \pi(\omega^{\sigma_{L/K'}})$. The condition $\pi^\vee = \pi^\sigma$ which one can also read $\pi(\omega^{-1}) = \pi(\omega^{\sigma_{L/K'}})$, is then equivalent from Appendix B, (2)b)1) of [G-L], to $\omega^{\sigma_{L/K'}} = \omega^{-1}$ or $\omega^{\sigma_{L/K''}} = \omega^{-1}$. This is equivalent to ω trivial on $N_{L/K'}(L^*)$ or on $N_{L/K''}(L^*)$.

2. L/F is cyclic, the regularity of ω makes the condition $\pi(\omega^{-1}) = \pi(\omega)^\sigma$ impossible. Indeed one would have from Theorem 1 of [G-L] $\pi(\omega^{\tilde{\sigma}}) = \pi(\omega^{-1})$, which from Appendix B, (2)b)1) of [G-L] would imply $\omega^{\tilde{\sigma}} = \omega$ or $\omega^{\tilde{\sigma}^{-1}} = \omega$. As $\tilde{\sigma}^2 = \tilde{\sigma}^{-2} = \sigma$, this would in turn imply $\omega^\sigma = \omega$, and ω would be trivial on the kernel of $N_{L/K}$ according to Hilbert's Theorem 90. $\pi^\vee$ can therefore not be isomorphic to π^σ.

3. L/K is not Galois (which implies $q \equiv 3[4]$ in the case p odd). Let $\pi_{B/K}$ be the representation of $G_2(B)$ which is the base change lift of π to B. As $\pi_{B/K} = \pi(\omega \circ N_{M/L})$, if $\omega \circ N_{M/L} = \mu \circ N_{M/B}$ for a character μ of B^*, then $\pi(\omega) = \pi(\mu)$ (cf.[G-L], (3) of Appendix B) and we are brought back to case 1. Otherwise $\omega \circ N_{M/L}$ is regular with respect to $N_{M/B}$. If $\pi^\sigma = \pi^\vee$, we would have $\pi_{B/K}^{\sigma_{B/K'}} = \pi_{B/K}^\vee$ from Theorem 1 of [G-L]. That would contradict case 2 because M/K' is cyclic.

□

We described in the previous proposition representations π of $G_2(K)$ verifying $\pi^\vee = \pi^\sigma$, now we characterize those who are $G_2(F)$-distinguished among them (from Theorem 4.2, a distinguished representation always satisfies the previous condition).

Theorem 4.4. *A dihedral supercuspidal representation π of $G_2(K)$ is $G_2(F)$-distinguished if and only if there exists a quadratic extension B of K biquadratic over F such that if we call K' and K'' the two other extensions between B and F, there is character ω of B^* that does not factorize through $N_{B/K}$ and trivial either on K'^* or on K''^*, such that $\pi = \pi(\omega)$.*

Proof. From Theorem 4.2 and Proposition 4.2, we can suppose that $\pi = \pi(\omega)$, for ω a regular multiplicative character of a quadratic extension B of K biquadratic over F, with ω trivial on $N_{L/K'}(K'^*)$ or on $N_{B/K''}(K''^*)$. We will need the following:

Lemma 4.1. *Let B be a quadratic extension of K biquadratic over F, then F^* is a subset of $N_{B/K}(B^*)$*

Proof of Lemma 4.1. The group $N_{B/K}(B^*)$ contains the two groups $N_{B/K}(K'^*)$ and $N_{B/K}(K''^*)$, which, as $\sigma_{B/K}$ extends $\sigma_{K'/F}$ and $\sigma_{K''/F}$, are respectively equal to $N_{K'/F}(K'^*)$ and $N_{K''/F}(K''^*)$. But these two groups are distinct of index 2 in F^* from local cassfield theory, thus they generate F^*, which is therefore contained in $N_{B/K}(B^*)$. □

We now choose ψ a non trivial character of K/F and denote by ψ_B the character $\psi \circ Tr_{B/K}$, it is trivial on K' and K''.
Suppose ω trivial on K' or K'', then the restriction of the central character $\eta_{B/K}\omega$ of $\pi(\omega)$ is trivial on F^* according to Lemma 4.1.
As we have $\gamma(\pi(\omega), 1/2, \psi) = \lambda(B/K, \psi)\gamma(\omega, 1/2, \psi_B) = \gamma(\eta_{B/K}, 1/2, \psi)\gamma(\omega, 1/2, \psi_B)$, we deduce from Lemma 4.1 and Proposition 4.1 that $\gamma(\pi(\omega), 1/2, \psi)$ is equal to one, hence from Theorem 4.1, the representation $\pi(\omega)$ is distinguished.
Now suppose $\omega|_{K'} = \eta_{B/K'}$ or $\omega|_{K''} = \eta_{B/K''}$, let χ be a character of K^* extending $\eta_{K/F}$, then $\pi(\omega) \otimes \chi = \pi(\omega\chi \circ N_{B/K})$. As $N_{B/K_{|K'}} = N_{K'/F}$ and $N_{B/K_{|K''}} = N_{K''/F}$, we have $\chi \circ N_{B/K_{|K'}} = \eta_{B/K'}$ and $\chi \circ N_{B/K_{|K''}} = \eta_{B/K''}$, hence from what we've just seen, $\pi(\omega) \otimes \chi$ is distinguished, i.e. $\pi(\omega)$ is $\eta_{K/F}$-distinguished.
From Theorem 4.3, π cannot be distinguished and $\eta_{K/F}$-distinguished at the same time, and the theorem follows. □

We end with the following lemma:

Lemma 4.2. *Let B be a quadratic extension of K which is biquadratic over F. Call K' and K'' the two other extensions between F and B, then the kernel of $N_{B/K}$ is a subgroup of the group $N_{B/K'}(B^*)N_{B/K''}(B^*)$.*

Proof. If u belongs to $Ker(N_{B/K})$, it can be written $x/\sigma_{B/K}(x)$ for some x in B^* according to Hilbert's Theorem 90. Hence we have $u = (x\sigma_{B/K'}(x))/(\sigma_{B/K}(x)\sigma_{B/K'}(x)) = N_{B/K'}(x)/N_{B/K''}(\sigma_{B/K}(x))$, and u belongs to $N_{B/K'}(B^*)N_{B/K''}(B^*)$. □

Corollary 4.1. *The (either/or) in Proposition 4.2 and Theorem 4.4 is exclusive*

Proof. In fact, in the situation of Lemma 4.2, a character ω that is trivial on $N_{B/K'}(B^*)$ and $N_{B/K''}(B^*)$ factorizes through $N_{B/K}$, and $\pi(\omega)$ is not supercuspidal. $\square$

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